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STRAIN DUE TO A CONTINUOUS, ISOTROPIC DISTRIBUTION OF SCREW DISLOCATIONS

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Abstract—The density for a continuous distribution of dislocations, represented by the torsion tensor, is separated into its two irreducible parts. One part characterizes an isotropic, homogeneous distribution of screw dislocations; the other, a mixture of screw and edge. A second order non-linear differential equation for the metric is established using the condition of zero affine curvature. In principle this equation can be solved for the metric once the distribution of dislocations is determined. The Robertson–Walker metric which describes an isotropic, homogeneous space is realized for an isotropic, homogeneous distribution of screw dislocations. The strain due to this distribution is then easily found from the simple relation between strain and metric.

1. INTRODUCTION

The aim of this paper is to present an exact solution for the strain due to a continuous, isotropic, homogeneous distribution of screw dislocations. In general, strain can be found once the geometry of the medium generated by dislocations is determined. This is because dislocation density can be represented geometrically by a torsion tensor, as shown by Bilby[1] and independently by Kondo[2]. Furthermore, strain itself is determined from a geometric object, the metric tensor.

In order to understand the geometry, it is important to consider the curvature of the medium. It is physically required that the lattice be uniquely defined even when dislocations are present; this leads to zero total curvature, represented by the affine curvature tensor. This curvature tensor can be considered as the sum of two tensors; the Riemannian curvature tensor which depends on second order non-linear differentials of the metric plus a tensor depending on torsion, i.e. dislocation density. The vanishing of the total curvature tensor yields second order non-linear differential equations for the metric, governed by the distribution of dislocations. In general, approximations are needed in order to solve the equations for the metric and hence the strain. For example, Eshelby[3] gives a solution for the strain when linearization of the equations is applicable.

In this paper we present an exact solution for the metric for the special case of an isotropic homogeneous distribution of screw dislocations. From this distribution the Riemannian part of the curvature tensor is found to have a simple form. Indeed, the metric leading to this particular Riemannian curvature has been well studied in general relativity for homogeneous, isotropic cosmologies. It is the so-called Robertson–Walker metric. From this metric, the expression for the strain is easily determined.

2. RELATION BETWEEN TORSION AND DISLOCATION DENSITY

The conventions used in this paper are such that all indices, upper and lower, run from one to three; repeated indices are summed over; comma denotes differentiation; indices surrounded by parentheses or square brackets indicate symmetric or antisymmetric components respectively.

As mentioned in the introduction, the density of a continuous distribution of dislocations is given by a torsion tensor, $C^{\alpha}_{\beta\gamma}$. Torsion can be pictured as a closure failure of a parallelogram constructed out of basis vectors, $\hat{e}_\alpha$, $\hat{e}_\beta$. In a space without torsion a closed parallelogram can be formed by taking two infinitesimal basis vectors located at the same point (Fig. 1a) and moving each one along the other parallel to itself. When torsion exists the parallelogram formed by this construction is not a closed quadrilateral, the gap being proportional to the torsion (Fig. 1b).

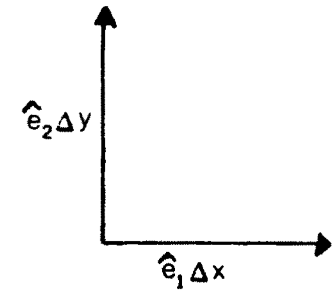

Fig. 1(a).

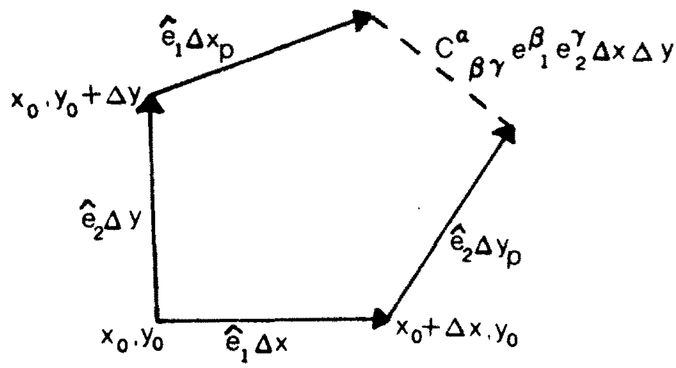

Fig. 1(b).

Fig. 1. (a) Two given infinitesimal basis vectors, $\hat{e}_1\Delta x$, $\hat{e}_2\Delta y$. (b) $\hat{e}_1\Delta x_p$ at the point x_0, $y_0+\Delta y$ and $\hat{e}_2\Delta y_p$ at the point $x_0+\Delta x$, y_0 are identified as parallel to $\hat{e}_1\Delta x$ and $\hat{e}_2\Delta y$ at x_0, y_0 respectively. A closed parallelogram is not formed from these four vectors. The gap in the parallelogram is proportional to the torsion, $C^\alpha_{\beta\gamma}$.

Algebraically, torsion is given as the antisymmetric part of the affine connection, $\Gamma^\alpha_{\beta\gamma}$:

$$C^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta} \tag{1}$$

where $\Gamma^\alpha_{\beta\gamma}$ is discussed further in Section 4.

A dislocation is determined by its Burgers vector and dislocation line. The first index of the torsion tensor, (1), gives the component of the Burgers vector in the x^α direction and the last two indices give the plane through which the dislocation line passes, $\hat{e}_\beta \wedge \hat{e}_\gamma$. There are two basic types of dislocations, screw and edge. For a screw dislocation, the Burgers vector and dislocation line are parallel, whereas for an edge, they are perpendicular. In general any dislocation can be considered a combination of edge and screw. In a solid crystal with dislocations, if vectors with the same crystallographic components are identified as parallel, a closure failure of parallelograms occurs; the gap is equal to the Burgers vector (Fig. 2). Comparing the gap in the parallelogram of Fig. 1 with those in Fig. 2 we see that torsion is essentially the Burgers vector per unit area.

3. THE IRREDUCIBLE PARTS OF THE TORSION TENSOR OR DISLOCATION DENSITY

As discussed by Toupin[4], any tensor of rank three has four irreducible parts. They are its symmetric part, antisymmetric part and two principal parts. For the torsion tensor, which is anti-symmetric in its last two indices, the symmetric and second principal part are zero. Thus, the surviving irreducible parts for the torsion tensor are its anti-symmetric part, ${}_AC_{\alpha\beta\gamma}$, and its

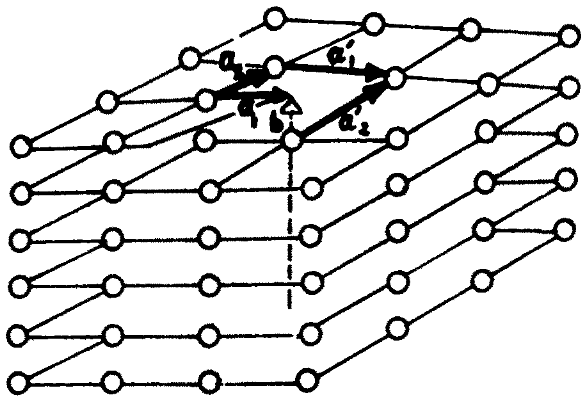

Fig. 2(a).

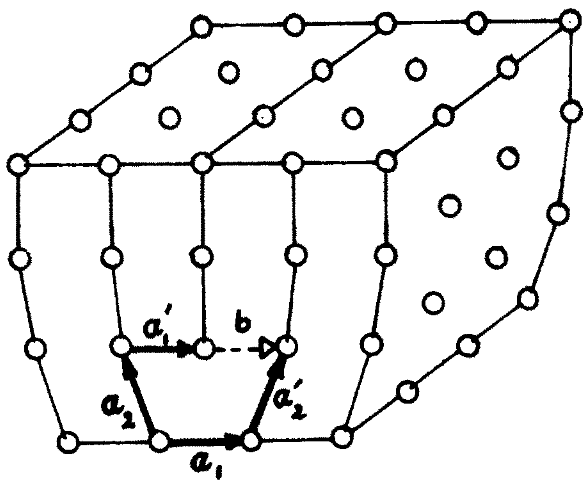

Fig. 2(b).

Fig. 2. (a) A screw dislocation. (b) An edge dislocation. **b** represents the Burgers vector; a_i $(i = 1, 2)$ represents crystallographic components. In each figure if a_1' is identified as parallel to a_1 and a_2' parallel to a_2, then the parallelogram formed by these vectors is not closed and the Burgers vector, **b**, can be related to the torsion; in fact, comparing with Fig. 1 we see that the Burgers vector per unit area is equal to the torsion.

first principal part, ${}_pC_{\alpha\beta\gamma}$

$$ {}_AC_{[\alpha\beta\gamma]} = C_{[\alpha\beta\gamma]} = \frac{1}{3}(C_{\alpha\beta\gamma} + C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta}) \tag{2}$$

$$ {}_pC_{\alpha\beta\gamma} = \frac{1}{3}(2C_{\alpha\beta\gamma} + C_{\beta\alpha\gamma} - C_{\gamma\alpha\beta}) = -{}_pC_{\alpha\gamma\beta} \tag{3}$$

$$ C_{\alpha\beta\gamma} = {}_AC_{\alpha\beta\gamma} + {}_pC_{\alpha\beta\gamma} \tag{4}$$

Note that we have lowered the first index of the torsion using the metric, $g_{\alpha\beta}$.

$$ C_{\alpha\beta\gamma} = g_{\alpha\lambda}C^{\lambda}_{\beta\gamma}. \tag{5}$$

The indices of any geometric object may be lowered or raised using the metric, $g_{\alpha\beta}$, or its inverse, $g^{\alpha\beta}$, e.g.

$$ \begin{aligned} A^{\lambda} &= g^{\lambda\nu}A_{\nu} \\ A_{\nu} &= g_{\nu\lambda}A^{\lambda}. \end{aligned} \tag{6}$$

The significance of $g_{\alpha\beta}$ is discussed in Section 4.

We can write ${}_AC_{\alpha\beta\gamma}$ uniquely in terms of the Levi–Civita tensor density, $\epsilon_{\alpha\beta\gamma}$, where

$$\epsilon_{\alpha\beta\gamma} = [\det(g_{\alpha\beta})]^{1/2}[\alpha\beta\gamma] \tag{7}$$

$$[\alpha\beta\gamma] \left\{ \begin{array}{ll} = +1 & \text{if } [\alpha\beta\gamma] \text{ is an even permutation of } 1,2,3 \\ = -1 & \text{if } [\alpha\beta\gamma] \text{ is an odd permutation of } 1,2,3 \\ = 0 & \text{if } [\alpha\beta\gamma] \text{ are not all different.} \end{array} \right. \tag{8}$$

Thus,

$$ {}_AC_{\alpha\beta\gamma} = M\epsilon_{\alpha\beta\gamma} \tag{9}$$

M has units or Burgers vector per unit area, i.e. reciprocal length. ${}_AC_{\alpha\beta\gamma}$ represents an isotropic, homogeneous distribution of screw dislocations. ${}_pC_{\alpha\beta\gamma}$ represents distributions of screw or edge dislocations, or both[5]. To see this, we form the dual dislocation density, $a_\alpha{}^\beta$, where

$$a_\alpha{}^\beta = C_{\alpha\rho\sigma}\epsilon^{\rho\sigma\beta}. \tag{10}$$

The first index in (10) represents the direction of the Burgers vector, and the second index the direction of the dislocation line. Since a screw dislocation has Burgers vector and dislocation line parallel, the two indices in (10) should match, and indeed, putting (9) into (10) we find

$${}_Aa_\alpha{}^\beta = 2M\delta_\alpha{}^\beta. \tag{11}$$

Taking the trace of (11) gives

$$M = \frac{1}{6}\, a_\alpha{}^\alpha. \tag{12}$$

So that

$${}_Aa_\alpha{}^\beta = \frac{1}{3}\, a_\lambda{}^\lambda \delta_\alpha{}^\beta. \tag{13}$$

Thus, ${}_Aa_\alpha{}^\beta$ represents a distribution of screw dislocations since it has matching indices. Furthermore, it is seen that the same density of dislocations, i.e. $2M$ or $(1/3)a_\lambda{}^\lambda$ is present in the 1, 2, 3 (or x, y, z) directions. This indicates that no distinction exists in these three directions, i.e. (9) or (13) represents an isotropic distribution. This is essentially because the Levi–Civita tensor, $\epsilon_{\alpha\beta\gamma}$, from which the dislocation density is constructed, is a handle-free tensor; i.e. it distinguishes no one direction. In addition, no particular points are distinguished by $\epsilon_{\alpha\beta\gamma}$, which implies homogeneity in the distribution. As seen in Section 5 this property of $\epsilon_{\alpha\beta\gamma}$ leads to a Riemannian curvature tensor for a homogeneous, isotropic space.

We now consider the significance of the other irreducible part of the dislocation density, ${}_pC_{\alpha\beta\gamma}$. Taking the dual of ${}_pC_{\alpha\beta\gamma}$ we find

$${}_pC_{\alpha\rho\sigma}\epsilon^{\rho\sigma\beta} = (2C_{\alpha\rho\sigma} + C_{\rho\alpha\sigma} - C_{\sigma\alpha\rho})\epsilon^{\rho\sigma\beta}. \tag{14}$$

Adding and subtracting $C_{\alpha\rho\sigma}$ to (14) gives

$$(C_{\alpha\rho\sigma} - {}_AC_{\alpha\rho\sigma})\epsilon^{\rho\sigma\beta} = a_\alpha{}^\beta - \frac{1}{3}\, a_\lambda{}^\lambda\delta_\alpha{}^\beta = {}_pa_\alpha{}^\beta. \tag{15}$$

An edge dislocation has its Burgers vector and dislocation line perpendicular so that the two indices in $a_\alpha{}^\beta$ must differ. Therefore an edge dislocation will be represented by the off-diagonal elements of ${}_Pa_\alpha{}^\beta$.

Although ${}_Pa_\alpha{}^\beta$ is a traceless tensor, diagonal elements (representing screw dislocations) are not excluded here. A general distribution of screw dislocations can be written as the sum of (13) and (15) in the dual representation[6]. For example, if the distribution has density M_1 in the 1 direction, M_2 in the 2 direction, and M_3 in the 3 direction, so that in matrix form

$$a_\alpha{}^\beta = \begin{pmatrix} M_1 & & 0 \\ & M_2 & \\ 0 & & M_3 \end{pmatrix} \tag{16}$$

(16) has elements in ${}_Aa_\alpha{}^\beta$ given by

$${}_Aa_\alpha{}^\beta = \begin{pmatrix} \frac{M_1+M_2+M_3}{3} & & 0 \\ & \frac{M_1+M_2+M_3}{3} & \\ 0 & & \frac{M_1+M_2+M_3}{3} \end{pmatrix} \tag{17}$$

and elements in ${}_Pa_\alpha{}^\beta$ given by

$${}_Pa_\alpha{}^\beta = \begin{pmatrix} M_1 - \frac{1}{3}(M_1+M_2+M_3) & & 0 \\ & M_2 - \frac{1}{3}(M_1+M_2+M_3) & \\ 0 & & M_3 - \frac{1}{3}(M_1+M_2+M_3) \end{pmatrix}. \tag{18}$$

In this more general case, there does not seem to be much gained by separating the distribution into its two irreducible parts.

To summarize, a general distribution of dislocations, $C_{\alpha\beta\gamma}$, has two irreducible parts: ${}_AC_{\alpha\beta\gamma}$ representing an isotropic, homogeneous distribution of screw dislocations and ${}_PC_{\alpha\beta\gamma}$, some elements of which represent edge dislocations, and others representing screw dislocations.

4. GEOMETRIC CONCEPTS

In order to find the strain due to a continuous isotropic homogeneous distribution of screw dislocations we first review pertinent geometric concepts. For further discussions, see any book on differential geometry, e.g. Bishop and Goldberg[7].

We begin with the affine connection mentioned in Section 2. The affine connection, $\Gamma^\alpha_{\beta\gamma}$, defines the notion of parallel transport. For a crystal solid, $\Gamma^\alpha_{\beta\gamma}$ has the form[3, 8]

$$\Gamma^\alpha_{\beta\gamma} = (\mu^{-1})_i{}^\alpha \mu^i_{\beta,\gamma} \tag{19}$$

where $\mu_\alpha{}^i$ are lattice correspondence functions that relate points in the dislocated crystal, X^i, to points in a perfect reference crystal, x^α. These lattice correspondence functions are analogous to the deformation gradient, $F_\alpha{}^i$, used in the theory of elasticity. There, $F_\alpha{}^i$ relates points in the deformed body, X^i, to points in a reference configuration, x^α, and has components

$$F_\alpha{}^i = \frac{\partial X^i}{\partial x^\alpha}. \tag{20}$$

See for example Marsden and Hughes[9]. By putting (19) into (2) we find the dislocation density (torsion tensor) in terms of $\mu_\alpha{}^i$

$$C^\alpha_{\beta\gamma} = (\mu^{-1})_i{}^\alpha(\mu^i_{\beta,\gamma} - \mu^i_{\gamma,\beta}). \tag{21}$$

Notice that if the lattice correspondence functions had the form of a gradient as in (20), then the dislocation density (21) would vanish since in this case $\mu^i[\beta, \gamma]$ would vanish identically.

We next introduce the curvature tensor, $R^\alpha_{\beta\gamma\delta}$. This gives the change in any vector, A^α, upon parallel transporting that vector around an infinitesimal closed path, i.e.

$$\Delta A^\alpha = R^\alpha_{\beta\gamma\delta}A^\beta\hat{e}^\gamma \wedge \hat{e}^\delta\Delta x\Delta y$$

where $\hat{e}^\gamma \wedge \hat{e}^\delta\Delta x\Delta y$ represents the infinitesimal area enclosed by the path. Algebraically, $R^\alpha_{\beta\gamma\delta}$ is given in terms of $\Gamma^\alpha_{\beta\gamma}$:

$$R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\lambda\gamma}\Gamma^\lambda_{\beta\delta} - \Gamma^\alpha_{\lambda\delta}\Gamma^\lambda_{\beta\gamma}. \tag{22}$$

If we put (19) into (22) we find

$$R^\alpha_{\beta\gamma\delta} = 0 \tag{23}$$

for a dislocated solid crystal.

Indeed, the form of the connection given by (19) was chosen precisely so that the curvature tensor vanishes. See for example Eisenhart[10]. This insures uniqueness of the lattice; i.e. lattice vectors are not rotated upon parallel transport about a complete circuit within the lattice.

Finally, we present the metric tensor, $g_{\alpha\beta}$. The metric gives the distance, ds, between any two points infinitesimally separated:

$$(ds)^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta. \tag{24}$$

Since the lengths of vectors are presumed to be conserved under parallel transport, the covariant derivative of the metric vanishes, i.e.

$$g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \Gamma^\lambda_{\alpha\gamma}g_{\lambda\beta} - \Gamma^\lambda_{\beta\gamma}g_{\alpha\lambda} = 0 \tag{25}$$

where semi-colon denotes covariant differentiation. (The covariant derivative of a tensor gives the change in that tensor due to parallel transport.) From (25) we find that the connection can be separated into two parts, $\mathring{\Gamma}^\alpha_{\beta\gamma}$ and $K^\alpha_{\beta\gamma}$, such that $\mathring{\Gamma}^\alpha_{\beta\gamma}$ contains only metric terms, and $K^\alpha_{\beta\gamma}$ only torsion terms.

$$\Gamma^\alpha_{\beta\gamma} = \mathring{\Gamma}^\alpha_{\beta\gamma} + K^\alpha_{\beta\gamma} \tag{26}$$

$$\mathring{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\lambda}(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda}) \tag{27}$$

$$K^\alpha{}_{\beta\gamma} = \frac{1}{2}(C^\alpha{}_{\beta\gamma} + C_{\beta\gamma}{}^\alpha + C_{\gamma\beta}{}^\alpha) = -K_\beta{}^\alpha{}_\gamma \tag{28}$$

In addition to (25) we find that the covariant derivative of the metric with respect to $\mathring{\Gamma}^\alpha_{\beta\gamma}$ vanishes as well, i.e.

$$g_{\alpha\beta/\gamma} = g_{\alpha\beta,\gamma} - \mathring{\Gamma}^\lambda_{\alpha\gamma}g_{\lambda\beta} - \mathring{\Gamma}^\lambda_{\beta\gamma}g_{\alpha\lambda} = 0 \tag{29}$$

where slash denotes covariant differentiation with respect to $\mathring{\Gamma}^\alpha_{\beta\gamma}$.

5. STRAIN

The metric is closely related to the Cauchy strain tensor, $\epsilon_{\alpha\beta}$, since

$$\epsilon_{\alpha\beta}\,dx^\alpha\,dx^\beta = \frac{1}{2}[(ds)^2 - (d\mathring{s})^2] \tag{30}$$

where $d\mathring{s}$ represents the distance between points infinitesimally separated in the perfect reference crystal and ds the distance between points in the dislocated crystal. For simplicity we can take

$$(d\mathring{s})^2 = \delta_{\alpha\beta}\,dx^\alpha\,dx^\beta. \tag{31}$$

In general,

$$(d\mathring{s})^2 = \mathring{g}_{\alpha\beta}\,dx^\alpha\,dx^\beta$$

where $\mathring{g}_{\alpha\beta}$ is the metric for a system of general curvilinear coordinates in the perfect crystal. The Cauchy strain tensor is then given by

$$\epsilon_{\alpha\beta} = \frac{1}{2}[g_{\alpha\beta} - \delta_{\alpha\beta}] \tag{32}$$

or

$$\epsilon_{\alpha\beta} = \frac{1}{2}[g_{\alpha\beta} - \mathring{g}_{\alpha\beta}]. \tag{33}$$

So if we know the metric of the dislocated crystal we can easily find the strain from (32) or (33).

If we put (26) into (22) and (23) we find

$$R^{\alpha}_{\beta\gamma\delta} = \mathring{R}^{\alpha}_{\beta\gamma\delta} + R'^{\alpha}_{\beta\gamma\delta} = 0 \tag{34}$$

where

$$\mathring{R}^{\alpha}_{\beta\gamma\delta} = \mathring{\Gamma}^{\alpha}_{\beta\delta,\gamma} - \mathring{\Gamma}^{\alpha}_{\beta\gamma,\delta} + \mathring{\Gamma}^{\alpha}_{\lambda\gamma}\mathring{\Gamma}^{\lambda}_{\beta\delta} - \mathring{\Gamma}^{\alpha}_{\lambda\delta}\mathring{\Gamma}^{\lambda}_{\beta\gamma} \tag{35}$$

$$R'^{\alpha}_{\beta\gamma\delta} = K^{\alpha}_{\beta\delta/\gamma} - K^{\alpha}_{\beta\gamma/\delta} + K^{\alpha}_{\lambda\gamma}K^{\lambda}_{\beta\delta} - K^{\alpha}_{\lambda\delta}K^{\lambda}_{\beta\gamma}. \tag{36}$$

The first term in (34), $\mathring{R}^{\alpha}_{\beta\gamma\delta}$, is the Riemannian part of the curvature tensor. It depends on first derivatives of $\mathring{\Gamma}^{\alpha}_{\beta\gamma}$ as well as on products of $\mathring{\Gamma}^{\alpha}_{\beta\gamma}$. But $\mathring{\Gamma}^{\alpha}_{\beta\gamma}$ depends on first derivatives of the metric (eqn 27). This results in a non-linear second order differential dependence of $\mathring{R}^{\alpha}_{\beta\gamma\delta}$ on the metric. Therefore (34) represents second order non-linear differential equations for the metric which, in principle, can be solved when $R'^{\alpha}_{\beta\gamma\delta}$ is determined. In general, approximations are needed in order to solve (34). However, we now derive an exact solution for the special case of an isotropic homogeneous distribution of screw dislocations.

The first step is to find $R'^{\alpha}_{\beta\gamma\delta}$. Putting $K^{\alpha}_{\beta\gamma} = (M/2)\epsilon^{\alpha}_{\beta\gamma}$ in (36) gives

$$R'^{\alpha}_{\beta\gamma\delta} = \frac{1}{2}[(M\epsilon^{\alpha}_{\beta\delta})_{/\gamma} - (M\epsilon^{\alpha}_{\beta\gamma})_{/\delta}] + \frac{M^2}{4}[\epsilon^{\alpha}_{\lambda\gamma}\epsilon^{\lambda}_{\beta\delta} - \epsilon^{\alpha}_{\lambda\delta}\epsilon^{\lambda}_{\beta\gamma}]. \tag{37}$$

Since $\epsilon^{\alpha}_{\beta\delta/\gamma} = 0$, (37) reduces to

$$R'^{\alpha}_{\beta\gamma\delta} = \frac{1}{2}[\epsilon^{\alpha}_{\beta\delta}M_{,\gamma} - \epsilon^{\alpha}_{\beta\gamma}M_{,\delta}] + \frac{M^2}{4}[\epsilon^{\alpha}_{\lambda\gamma}\epsilon^{\lambda}_{\beta\delta} - \epsilon^{\alpha}_{\lambda\delta}\epsilon^{\lambda}_{\beta\gamma}]. \tag{38}$$

We now show that the first term in brackets in (38) is zero. Consider the contracted curvature tensor, $R_{\beta\delta}$, where

$$R_{\beta\delta} = R^{\alpha}_{\beta\alpha\delta}. \tag{39}$$

From (34),

$$R_{\beta\delta} = \mathring{R}_{\beta\delta} + R'_{\beta\delta} = 0 \tag{40}$$

which implies

$$R_{[\beta\delta]} = \mathring{R}_{[\beta\delta]} + R'_{[\beta\delta]} = 0. \tag{41}$$

But $\mathring{R}_{\beta\delta}$ is a symmetric tensor, i.e. $\mathring{R}_{[\beta\delta]} = 0$. Therefore,

$$R_{[\beta\delta]} = R'_{[\beta\delta]} = 0. \tag{42}$$

Using (38),

$$R'_{\beta\delta} = R'^{\alpha}_{\beta\alpha\delta} = \frac{1}{2}\epsilon^{\alpha}_{\beta\delta}M_{,\alpha} - \frac{1}{4}M^2\epsilon^{\alpha}_{\lambda\delta}\epsilon^{\lambda}_{\beta\alpha}. \tag{43}$$

Putting (43) in (42) yields

$$R'_{[\beta\delta]} = \frac{1}{2}\epsilon^{\alpha}_{\beta\delta}M_{,\alpha} \tag{44}$$

(44) holds only if

$$M_{,\alpha} = 0. \tag{45}$$

From (45) we see that the first term in (38) vanishes. Therefore

$$R'^{\alpha}_{\beta\gamma\delta} = \frac{M^2}{4}(\epsilon^{\alpha}_{\lambda\gamma}\epsilon^{\lambda}_{\beta\delta} - \epsilon^{\alpha}_{\lambda\delta}\epsilon^{\lambda}_{\beta\gamma}). \tag{46}$$

The first index in (46) may be lowered by using the metric:

$$R'_{\alpha\beta\gamma\delta} = g_{\alpha\sigma}R'^{\sigma}_{\beta\gamma\delta} = \frac{M^2}{4}(\epsilon_{\alpha\lambda\gamma}\epsilon^{\lambda}_{\beta\delta} - \epsilon_{\alpha\lambda\delta}\epsilon^{\lambda}_{\beta\gamma}). \tag{47}$$

Since

$$\epsilon_{\lambda\alpha\delta}\epsilon^{\lambda}_{\beta\gamma} = g_{\alpha\beta}g_{\delta\gamma} - g_{\alpha\gamma}g_{\beta\delta} \tag{48}$$

(47) can be written as

$$R'_{\alpha\beta\gamma\delta} = \frac{M^2}{4}(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}). \tag{49}$$

Thus, (34) gives

$$\mathring{R}_{\alpha\beta\gamma\delta} = \frac{M^2}{4}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}). \tag{50}$$

The form of the Riemannian curvature, $\mathring{R}^{\alpha}_{\beta\gamma\delta}$, given by (50) is well known in the theory of general relativity. It represents an isotropic homogeneous space of constant positive curvature. (The sign of the curvature is determined from the sign of the coefficient of the term in parentheses in (50).) The metric satisfying (50) was first found by Robertson[11] and Walker[12] and is named after them. In terms of the coordinates 1, 2, 3, it is given as

$$g_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{\left(1 + \frac{M^2}{16}\,\delta_{\rho\sigma}X^\rho X^\sigma\right)^2}. \tag{51}$$

The infinitesimal line element squared, derived from (51) is then

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{\left(1 + \frac{M^2}{16}(x^2 + y^2 + z^2)\right)^2}. \tag{52}$$

By putting (51) into (32) the strain is found to be

$$\epsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{1}{(1 + (M^2/16)\delta_{\rho\sigma}X^\rho X^\sigma)^2} - 1\right)\delta_{\alpha\beta} \tag{53}$$

(53) represents an exact expression for the strain generated by a continuous isotropic homogeneous distribution of screw dislocations.

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5. I am grateful to the referee for suggesting a misinterpretation of this point in the first version of this paper. There it was stated that ${}_AC^{\alpha}_{\beta\gamma}$ represented screw dislocations and ${}_PC^{\alpha}_{\beta\gamma}$ represented edge dislocations exclusively.
6. This statement stems from the referee's comment on the first version of this paper.
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